OFFICE OF THE REGISTRAR OF BOOKS, 7 JUL 1913, MADRAS.

THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. V.] JUNE 1913. [No. 3.

PROGRESS REPORT.

The following gentlemen have been elected as members of the Society :—

(1) Mr. K. K. Ranganatha Aiyar, M.A., Assistant Professor of Mathematics, St. Joseph's College, Mellachinthamoni, Teppakulum ;

(2) Mr. T. P. Bhaskara Shastri, B.A., Assistant, Nizamiah Observatory, Begumpet, Secundrabad, Deccan, (at concessional rate).

The following books has been received as presents to the Library—

(1) *Cours de Geometrie Infinitesimale*—by Professor G. Demartres, with a preface by Mon. P. Appell, (Gauthier-Villars), Paris, 1913, 17 frs.

(2) *Matrices and Determinoids*, Vol. I—by Prof. C. E. Cullis, M.A., Ph. D., Camb. Univ. Press, 1913, 21s/—.

POONA,
31*st May* 1913.

D. D. KAPADIA,
Hony. Joint Secretary.

The three-cusped Hypocycloid and its Spherical Analogue.

By M. T. Naraniengar.

[This paper, drawn up some years ago, had got mixed up with other papers of the author and was recovered only recently. Dr. Duncan M. Y. Sommerville, submitted to the Fifth International Congress of Mathematicians, Cambridge, a paper entitled '*The Pedal line of the Triangle in non-euclidean Geometry*' in August last, which may be read in connection with the latter part of the present paper.]

The Hypocycloid.

1. To find the trilinear equation of the three cusped hypocycloid referred to the cuspidal triangle.

The equation must be of the form

$$l\Sigma(\alpha^3\beta)+m\Sigma(\alpha^2\beta^2)+n\alpha\beta\gamma\Sigma(\alpha)=0,$$

since the curve is a quartic circumscribing the triangle of reference. Put $\beta=\gamma$, then the coefficient of β and β^2 must be zero for a cusp at A Hence, $l=o$, $n=-2m$.

Thus, the equation becomes

$$\Sigma(\alpha^2\beta^2)-2\alpha\beta\gamma\Sigma(\alpha)=0.$$

This is easily reduced to the form

$$\alpha^{-\frac{1}{2}}\pm\beta^{-\frac{1}{2}}\pm\gamma^{-\frac{1}{2}}=0,$$

which is therefore the equation required.*

2. *The three-cusped hypocycloid is the isogonal transformation of the in-circle.*

This follows at once, since the equation to the incircle is $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}=0$.

3. *To find the intersections of the hypocycloid and the incircle.*

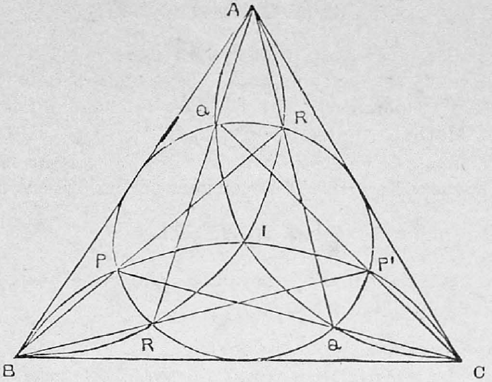
These are obviously three pairs of isogonal conjugates, and are the points of contact of tangents from the excentres to the incircle. For, we

* When referred to the triangle formed by the tangents at the vertices, the equation reduces to

$$(7\alpha+\beta+\gamma)^{-\frac{1}{2}}+\quad .\quad .\quad .\quad =0,$$

as may easily be verified.

know from Q. 39 (J. I. M. S.) that the pole of the chord joining two conjugate points on the incircle must be an excentre. (*Vide*: also, Q. 67).



It is further seen that these six points are the vertices of two equilateral triangles PQR, P′Q′R′, whose sides are concurrent with the vertices of ABC. They may also be determined as the intersections of the incircle and the circles touching any two sides at their junctions with the third side.

4. *To find the tangential equation to the hypocycloid.*

The condition that the straight line $\xi\alpha+\eta\beta+\zeta\gamma=0$ shall touch the curve is readily seen to be

$$\xi : \eta : \zeta = \alpha'^{-\frac{3}{3}} : \beta'^{-\frac{3}{2}} : \gamma'^{-\frac{3}{2}}$$

where $(\alpha'\beta'\gamma')$ is a point on the curve, (*i.e.*) $\alpha'^{-\frac{1}{2}}+\beta'^{-\frac{1}{2}}+\gamma'^{-\frac{1}{2}}=0.$

Hence, the condition of tangency becomes

$$\xi^{\frac{1}{3}}+\eta^{\frac{1}{3}}+\zeta^{\frac{1}{3}}=0$$

which is the tangential equation required.

Cor. The polar reciprocal with respect to $\alpha^2+\beta^2+\gamma^2=0$ is thus $\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=0$, being identical with the condition that $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$ should touch the hypocycloid.

5. The equation $\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=0$ may be reduced to the polar form $r \cos 3\theta=h$, where h is altitude of the triangle of reference.

For, if we put

$$\alpha=h/3+r\cos\theta$$
$$\beta=h/3+r\cos 120^\circ+\theta$$
$$\gamma=h/3+r\cos 240^\circ+\theta$$

$\alpha+\beta+\gamma=h$, and the equation $\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=0$ reduces to

$$(\alpha+\beta+\gamma)^3=27\alpha\beta\gamma\,;$$

that is, $h^3=27\,(h/3+r\cos\theta)(h/3+r\cos 120^\circ+\theta)\,(h/3+r\cos 240^\circ+\theta)$.

Simplifying

$$h^3=27[h^3/27+hr^2/3(-\tfrac{3}{4})+r^3\cos3\theta/4]$$

or

$$r\cos3\theta=h.$$

This may also be inferred from the fact that the polar reciprocal is the inverse of the pedal of the hypocycloid.

6. Any point on the hypocycloid will be denoted by (x,y,z), where (x^{-2},y^{-2},z^{-2}) are the trilinear coordinates of the point. Thus, if (xyz) is a point on the curve $x+y+z=0$ and the tangent at the point is written in the form $\alpha x^3+\beta y^3+\gamma z^3=0$.

Ex. 1. *If APQ is a chord through the vertex A, to find the locus of the intersection of the tangents at P & Q.* (*Vide*: Basset's *Cubic and Quadric Curves*, p. 243).

Let P be the point (x,y,z), then Q is seen to be the point $(y-z,-y,z)$, since APQ is a straight line. The tangents at P & Q are

$$\alpha x^3+\beta y^3+\gamma z^3=0 \text{ and } \alpha\,(y-z)^3-\beta y^3+\gamma z^3=0.$$

Solving for $\alpha:\beta:\gamma$, we get

$$\alpha:\beta:\gamma=2y^3z^3:\ z^3\{(y-z)^3-x^3\}:\ y^3\{-x^3-(y-z)^3\}$$
$$=2y^3z^3:\ 2z^3(y^3+3yz^2):\ 2y^3(3y^2z+z^3)$$
$$=1:\ (1+3z^2/y^2):\ (1+3y^2/z^2)$$

Hence, $(\beta-\alpha)(\gamma-\alpha)=9\alpha^2$ is the locus required.

Ex. 2. *Any staight line $l\alpha+m\beta+n\gamma=0$ cuts the hypocycloid in four points $(x\ y\ z)$; to prove that $x_1x_2x_3x_4$ varies as l.*

The ratios $x:y:z$ are given by

$$l/x^2+m/y^2+n/z^2=0 \text{ where } x+y+z=0.$$

$$\therefore\quad ly^2z^2+mz^2(y+z)^2+ny^2(y+z)^2=0$$

$$\therefore\quad y_1y_2y_3y_4:\ z_1z_2z_3z_4=m:n. \quad \text{Hence the result.}$$

Cor. If the tangent at (x,y,z) meet the curve again at $(x_1y_1z_1)$ $(x_2y_2z_2)$ we have

$$l:m:n=x^3:y^3:z^3 \text{ and } x_3=x_4=x \text{ \&c.}$$

$$\therefore\quad x_1x_2:\ y_1y_2:\ z_1z_2=x:y:z.$$

In other words,

$$x_1x_2+y_1y_2+z_1z_2=0,$$

and the trilinear coordinates of the point of contact vary as the products of the trilinear coordinates of the points of intersection.

Further, this equation together with the conditions $x_1+y_1+z_1=0$; $x_2+y_2+z_2=0$ will serve to shew that the tangents at the points $(x_1\,y_1\,z_1)$ $(x_2y_2z_2)$ intersect at right angles on a circle.

Ex. 3. *If tangents be drawn from any point* (l, m, n) *to prove that* $\Sigma(x_1\, x_2\, x_3)=0$.

The tangent at (xyz) is

$$\alpha x^3+\beta y^3+\gamma z^3=0.$$

If this passes through (l, m, n) we have $lx^3+my^3+nz^3=0$, with the condition $x+y+z=0$.

Hence, $\quad l\,(y+z)^3=my^3+nz^3$

$\therefore \quad y_1y_2y_3 : z_1z_2z_3=(n-l) : (l-m)$.

Thus $\quad x_1x_2x_3+ \quad \ldots \quad \ldots \quad =0$.

Conversely: if $\Sigma\,(x_1x_2x_3)=0$, the tangents at $(x_1y_1z_1)$ &c., are concurrent. For, the point $(x_3y_3z_3)$ is uniquely determined by the equations

$$\Sigma\,(x_1x_2x_3)=0,\ \Sigma\,(x_3)=0.$$

Ex. 4. *If* $(x_1y_1z_1)$ *&c, be the points of contact of three concurent tangents, then the tangents from these points of contact are also concurrent.*

For, let $(X_1\,Y_1\,Z_1)$ be the point of contact of the tangent from $(x_1y_1z_1)$, then by taking $l : m : n$ (in Ex. 3) $=\dfrac{1}{x_1^2} : \dfrac{1}{y_1^2} : \dfrac{1}{z_1^2}$ and putting $x_1=x_2$ & $x_3=X_1$ &c., we find

$$X_1.\ x_1^2,\ \text{varies as} \left(\frac{1}{z_1^2}-\frac{1}{y_1^2}\right)$$

$$\therefore \quad X_1 : Y_1 : Z_1=y_1^2-z_1^2 : z_1^2-x_1^2 : x_1^2-y_1^2.$$

Similarly for the other points.

$$\begin{aligned} \text{Now, } X_1\,X_2\,X_3 &\propto (y_1^2-z_1^2)(y_2^2-z_2^2)(y_3^2-z_3^2) \\ &\propto x_1x_2x_3.\ z_1z_2z_3\left(1-\frac{y_1}{z_1}\right)\left(1-\frac{y_2}{z_2}\right)\left(1-\frac{y_3}{z_3}\right) \\ &\propto (m-n)\,(l-m).(8l-m-n)/(l-m) \\ &\propto 8l\,(m-n)-(m^2-n^2) \end{aligned}$$

Hence, $\quad \Sigma\,(X_1\,X_2\,X_3)=0$, and therefore by the converse of Ex. 3 the tangents at $(X_1\,Y_1\,Z_1)$ &c., are concurrent.*

Further, since concurrent normals may be looked upon as tangents to the evolute, which is another three casped hypocycloid, the property is true of normals also.*

[*Note*:—The points of concurrence are interdepenlant, and when the second describes a straight line the first describes a conic].

7. Consider the intersection of the tricusp with the

conic $\quad \dfrac{l}{\alpha}+\dfrac{m}{\beta}+\dfrac{n}{\gamma}=0.$

---

* This property is easily extended to the cardioide by projection. [*Vide* Q. 16778, *Ed. Times*.]

$\therefore \qquad lx^2+my^2+nz^2=0.$

$\therefore \qquad x_1x_2 : y_1y_2 : z_1z_2 = m+n : n+l = l+m$

If $l+m+n=0$, the conic passes through the ortho-centre and thus we have the theroem :

All rectangular hyperbolas circumscribing the cuspidal triangle meet the tricusp in points P, Q. such that PQ is a tangent to the hypocycloid.

The following theorems can be established :

(1) *The asymptotes of the hyperbola are parallel to the tangents at* **P**, *Q to the tricusp and* (2) *the centre of the hyperbola is the isogonal conjugate of the point of contact of PQ with the tricusp.*

(1) Taking $\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0$, with $l+m+n=0$, the asymptotes are

$$\Sigma(l\beta\gamma)\ (l^2+m^2+n^2-2lm-2ln-mn)+2lmn\ (\alpha+\beta+\gamma)^2=0,$$

which reduces to $\Sigma(l\beta\gamma).(l^2+m^2+n^2)+lmn(\alpha+\beta+\gamma)^2=0.$

The tangents at $(x_1y_1z_1)(x_2y_2z_2)$ are

$$(\alpha x_1^3+\beta y_1^3+\gamma z_1^3)(\alpha x_2^3+\beta y_2^3+\gamma z_2^3)^3=0$$

and $\qquad x_1x_2 : y_1y_2 : z_1z_2 = l : m : n.$

Thus $\qquad x_1^3x_2^3=l^3,\ x_1^3y_2^3+x_2^3y_1^3=2n\ \{\ 4n^2-lm\ \},$

since $x_1\,y_2+x_2\,y_1=z_2z_1-x_1x_2-y_1y_2=2n.$

Thus the tangents are

$$l^3\alpha^2+m^3\beta^2+n^3\gamma^2+8(l^3\beta\gamma+m^3\gamma\alpha+n^3\alpha\beta)-6\ lmn(\beta\gamma+\gamma\alpha+\alpha\beta)=0.$$

These will be parallel to the asymptotes if it is possible to choose k so that when $k(l^2+m^2+n^2)\ (l\beta\gamma+m\gamma\alpha+n\alpha\beta)$ is added, the result is divisible by $\alpha+\beta+\gamma$ and will be in fact

$$(l^3\alpha+m^3\beta+n^3\gamma)\ (\alpha+\beta+\gamma).$$

Hence $\qquad l^3+m^3=k(l^2+m^2+n^2)\ n+8n^3-6\ lmn.$

$$\therefore \{\ k(l^2+m^2+(l+m)^2\ \}=-8(l+m)^2+6\ lm.\ (l^2-lm+m^2)$$
$$=-9\ l^2-9\ lm-9m^2,$$

whence $\qquad k=-\frac{9}{2}.$

(2) The centre of $\Sigma(l\beta\gamma)=0$ is given by

$$\frac{\alpha}{l(l-m-n)}=\ \dots\ =\ \dots$$

and is therefore (l^2, m^2, n^2), the point of contact is $(1/l^2, 1/m^2, 1/n^2)$.

Thus the centre is the isogonal conjugate of the point of contact of the tangent.*

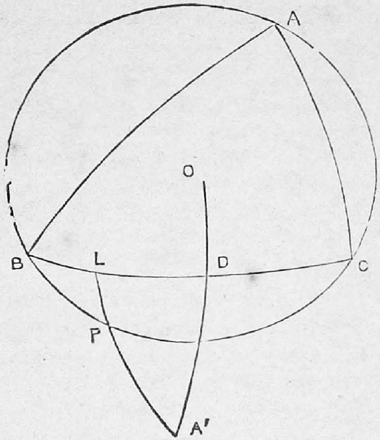
---

* The theorems of § 7 were kindly communicated to me by Prof. A C. L. Wilkinson.

The Spherical Analogue.

8. Let ABC be a spherical triangle. Take a point P on the circum-circle. Then we know that the feet of the perpendiculars from P on the sides of the chordal triangle are collinear (Simson-line property), and that the envelope of the *pedal line* is a three-cusped hypocycloid.

Now, the circumcentre of the chordal triangle projects into the pole (O) of the circum circle of the spherical triangle ABC ; the mid-point of the chord BC projects into the mid-point of the great circle arc BC,



Thus, the diameter of the chordal triangle $\perp^r$ to the chord BC projects into the arc through O bisecting the arc BC. Also, a chord through P parallel to this diameter projects into an arc through P meeting OD produced on the *polar circle* of O.

Hence, the projections of the $\perp^r$ from P on the sides of the chordal triangle are PL, PM, PN, such that PL, OD ; PM, OE ; PN, OF meet on the polar great circle of O. Further, the points L, M, N being the projections of three collinear points lie on a great circle. In other words, we have the following property :—If OD, OE, OF be produced to A′, B′, C′ so that $OA'=OB'=OC'=\pi/2$, the points of intersection of PA′, PB′, PC′ with the sides of ABC are collinear, and the line of

collinearity may be called the *pedal* of the point P with respect to the spherical triangle ABC.

9. Since the envelope of the Simson line of P for different positions of P on the circumcircle is a tricusp, it follows that the spherical envelope of the pedal line of P is also a tricusp, being the projection of the former on the sphere. This tricusp is readily seen to touch the sides of the spherical triangle ABC, since these sides are the *pedals* of points of the small circle diametrically opposite to A, B, C.*

The tricusp also touches the isogonal conjugates of AO, BO, CO, with respect to A, B, C respectively. For, it will be seen that the projections of the $\perp^{rs}$ of the chordal triangle are the isogonal conjugates of AO, BO, CO.

Thus we have proved that the *envelope of the pedal line, P is a tricusp touching the sides of ABC, as well as the isogonal conjugates of AO, BO, CO*.

10. In the particular case of an equilateral spherical triangle ABC, the tricusp is symmetrical with respect to O, and its equation in *normal* coordinates may easily be deduced.

For, the equation to the circumcircle is identical *in form* with that in plano, *viz.*, $x^{-1}+y^{-1}+z^{-1}=0$. And, the equations to OD, PL, &c., are also identical *in form* since the *plane condition* of parallelism is the same as the spherical condition that lines shall intersect on the polar circle of O (*vide*: Prog. Rep.: 8, November 1908). Therefore, the equation to the pedal of *P* is *identical in form* with that *in plano*. Hence, also, the equation of the envelope is the *same in form* when referred to the cuspidal triangle; its equation must therefore be $x^{-\frac{1}{2}}+y^{-\frac{1}{2}}+z^{-\frac{1}{2}}=0$.

11. The tangential equation of the equilateral tricusp may be written, as in plano, $\xi^{\frac{1}{3}}+\eta^{\frac{1}{3}}+\xi^{\frac{1}{3}}=0$.

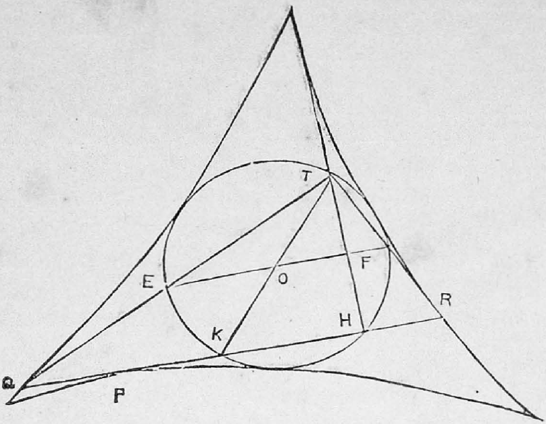
12. The following properties are remarkable.

(i) The tangents at the ends of a tangent chord QPR intersect at T on the small circle inscribed within the tricusp.

---

* The Simson-lines of these points are the chords BC, &c., and therefore their spherical *pedals* are the sides of the *spherical* triangles ABC.

(ii) The other points of intersection of the tangents (E, F) are the ends of a diameter of the small circle.



(iii) The third tangent from T meets QR on the small circle.

(iv) The intersection of TO and QR lies on the small circle.

All these properties are proved at once by observing that the coordinates of the several points, and the equations to the several times in the figure are the same in form as in plano, and that therefore all *concurrent* and *linear* properties in plano are true for the sphere also.

Further, EF and QR meet on the polar circle of O, since the corresponding lines *in plano* are parallel (*vide* : § 7.)

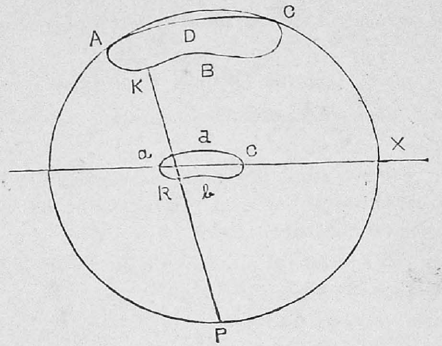
13. All properties of the hypocycloid depending upon linear and coordinate properties may easily be extended to this spherical tricusp Thus, it will be seen that Examples 1—4 of § 6 apply to the sphere also.

Application of Stereographic Projection to Astronomical Problems.

By T. K. Venkataraman.

Let ABCD be any curve drawn on the surface of a sphere, X any plane through the centre of the sphere, and P the pole of the great circle in which the plane X cuts the sphere. If K be a variable point on the curve, and PK be joined to cut the X-plane in k, the locus of k as K traces the curve ABCD is called the stereographic projection of the curve ABCD on the plane X. P is called the centre of projection.

Fig. 1.



2. The following properties are assumed. The proofs may be found in Casey's or McLelland and Preston's Spherical Trigonometry :

(1) The stereographic projection of a circle great or small, the plane of which does not pass through the centre of projection, is a circle.

(2) Circles whose planes pass through the centre of projection project into right lines.

(3) Circles having the centre of projection for pole, project into concentric circles.

(4) The angle of intersection of any two spherical curves, is equal to the angle of intersection of their projections at the corresponding points.

3. Many Astronomical problems depend for their solution on finding the angles between circles, great or small, on the celestial sphere. Such problems can be practically solved by drawing the projections of those circles on a plane conveniently chosen, and measuring the required

angles in the projection, The most convenient planes to project on are (1) the equator, (2) the horizon, and (3) the meridian.

4. The projection of any given circle can be drawn readily by first noting the projections of three convenient points on it, and drawing a circle to pass through those projections.

5. Considerations of symmetry also help us to draw the projection of a circle: *e.g.* the diurnal circles of all stars being symmetrical about the meridian, their projections on the horizontal plane are symmetrical about the straight line into which the meridian projects. In Figs. 2 and 3, $x\sigma$ is the projection of the star's daily path and is symmetrical about QpQ′, the projection of the meridian; also in Fig. 12 ♈k♎ is the projection of the ecliptic, and is symmetrical about pk, which is the projection of a central plane in the celestial sphere perpendicular to ♈ ♎.

Some Typical Problems.

I. *To find the hour angle of a known star when it rises at a given place.*

Since the required angle is the angle between the meridian and the declination circle through the star, we must obtain the projections of these two circles on a suitable plane.

Project on the equatorial plane, with the South Pole as the centre of projection. Figs. 2 and 3.

The equator projects into itself.

The meridian projects into a diameter QPQ′ of the equator, § 2 (2).

The diurnal circle of the star projects into a circle concentric with the equator; § 2 (3).

To find the radius of this concentric circle: join P′Σ, where P′ is the South Pole, and Σ is the point where the star crosses the meridian; Fig. 2.

If P′Σ cuts QQ′ in σ, $p\sigma$ is the required radius.

Thus ⊙ σx, Fig 3, is the projection of the daily path.

To find the projection of the horizon: let N and S be the North and South points, Fig. 2.

Join P′N, P′S cutting the straight line QQ′ at n and s. From symmetry, the ⊙ on ns as diameter is the projection of the horizon, Fig 3.

x, the point of intersection of the projections of the daily path and the horizon, is the projection of the position of the star at rising.

The projection of any declination ⊙ is a straight line, § 2 (2)..

px, Fig. 3, is therefore the projection of the declination ⊙ of the star.

Hence the hour angle of the star at rising is the angle between *ps* and *px* § 2 (4).

Conversely : if the hour angle of a star at rising be given, we can find, by retracing the above steps, the declination of the star.

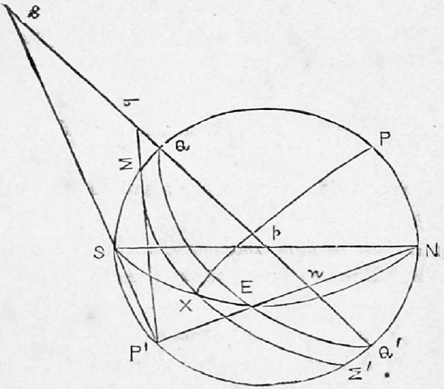
N. B. Finding the length of the day on a given date, and the converse problem, are only particular cases of the above.

This class of problems could be solved also by projecting on the horizontal plane.

Ex. 1. Find the sidereal time of rising of *Spica*, R. A. 13 hours 19 minutes, decl. $-10° 32'$, in lat. 45° N.

Fig. 2 represents the celestial sphere of the observer, with the usual lettering. and PN = 45°.

Fig. 2.



Mark off QΣ = 10° 32′, the S. declination of *Spica*. Fig 3. is the projection on the equatorial plane, with P′ as centre of projection.

Circle QEQ′ is the projection of the Equator,

QpQ′ „ Meridian,

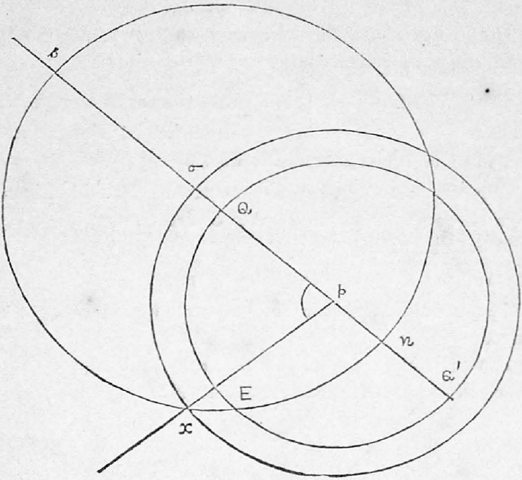
*n*E*s*′ „ Horizon.

The circle with p for centre and radius $p\sigma$ is the projection of the star's daily path.

x_2 the intersection of the horizon and the daily path, gives the position the of star at rising.

px is the projection of the declination circle of the star.

Fig. 3.



Hour ∠ at rising = ∠ between the meridian and PX in Fig. 2.

$= \angle\sigma px$ in Fig. 3.

$= 79^\circ\ 10' = 5$ h. 16 m. 40 s.

∴ Sid. time of rising = 13 h. 19 m.

− 5 h. 16 m. 40 s.

= 8 h. 2 m. 20 s.

II. *To find the hour angle and altitude of a known star when it is due East.*

Project on the horizontal plane, the nadir Z' being the centre of projection. Figs. 4 and 5.

Σ, Σ' being as before the points where the star transits the meridian join $Z'\Sigma$, $Z'\Sigma'$ cutting NS at σ, σ'.

From symmetry, the ⊙ $\sigma x \sigma'$ on $\sigma\sigma'$ as diameter is the projection of the daily path of the star, Fig. 5.

The horizon projects into itself.

The meridian projects into the straight line NS, § 2 (2).

The prime vertical projects into the straight line WZE, perpendicular to NZS, §§ 2(2) and 2(4).

The intersection of ⊙ $\sigma x \sigma'$ and straight line WZE gives the projection of the position of the star when due east.

The ⊙ through p, p' and x, is therefore the projection of the declination ⊙ of the star, § 4.

The required hour ∠ is the angle between straight line pS and ⊙ pxp'.

The altitude is the complement of the arc of the prime vertical which corresponds to the segment Zx, Fig. 5.

To find this arc, take the prime vertical EZWZ′, Fig. 6, where E,W are the east and west points, and Z,Z′ are the zenith and nadir.

Cut off $zx = zx$ from Fig. 5.

Join $Z'x$ meeting the prime vertical at x.

X is the position of the star on the prime vertical and ∠ZzX is the zenith distance.

Ex. 2.—Find the hour angle and altitude of *Procyon* (decl. 5° 32)′ when it crosses the prime vertical in lat. 48°N.

Fig. 4.

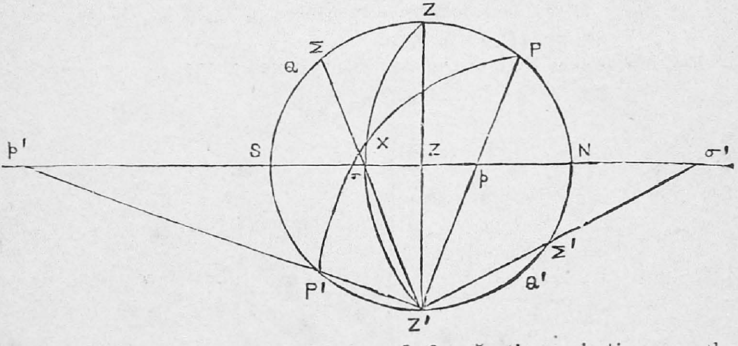
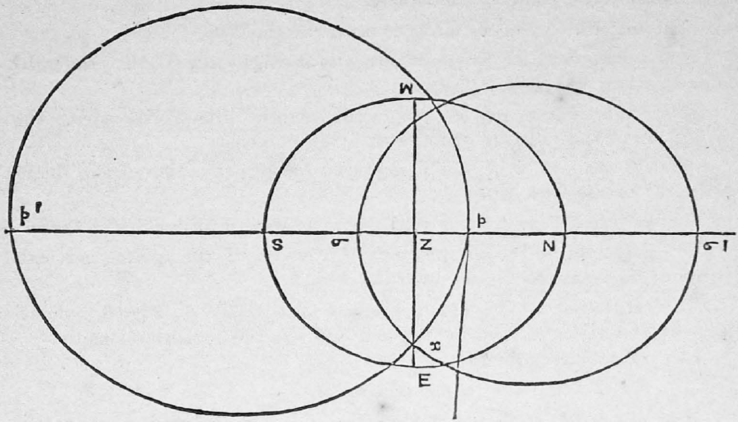


Fig. 4 is the celestial sphere, and fig. 5. the projection on the horizon. Here Σ,Σ′ are the points where the star crosses the meridian, σ,σ′ their projection on the horizon. ZxZ' is the prime vertical, and PxP' the decl. ⊙ of the star.

Fig. 5.



In Fig. 5 NESW is the horizon ; ⊙ on $\sigma\sigma'$ as diam. is the projection of the star's daily path ; WZE is the projection of the prime vertical ; the intersection of ⊙ $\sigma x\sigma'$, and WE gives the position of the star ; ⊙ pxp' is the projection of the star's decl. ⊙.

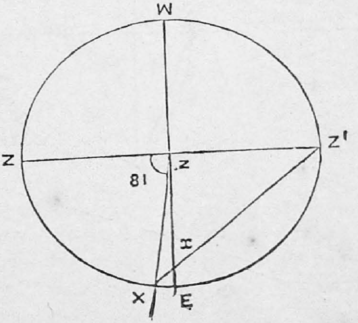
Hour ∠ of the star = ∠ between the meridian and ⊙ PxP' Fig. 4.

= ∠ between pzp' and circle pxp

$= 84\frac{3}{4}° = 4$ h. 35 m.

Fig. 6 gives the Z.D. of the star as explained above.

Fig. 6.



Z.D. $=81°$ $\therefore$ altitude $=9°$.

III. *To find the interval of time in which the altitude of a known star changes from one given value to another.*

The required interval is measured by the angle between the decl. ⊙ through the initial and final positions of the star.

Projecting on the horizontal plane, the projections of the horizon and the star's daily path should be obtained as before. Figs. 7 & 8.

If KK′ be a ⊙ parallel to the horizon whose altitude is NK (negative in Fig. 7), its projection on the horizontal plane is a ⊙ concentric with the horizon and whose radius is *zk*.

⊙ $k\sigma_1$ is therefore its projection, Fig. 8.

The points σ_1, σ_2 where the projection $\sigma\sigma_1\sigma^1$ of the daily path cuts the circle $k\sigma_1$ and the horizon, respectively are the projections of the two positions of the star.

$\therefore$ the ⊙ through $p\sigma_1$ p' and $p\sigma_2$ p' are the projections of the decl. circles.

N. B.—The problem of finding the duration of twilight on a given day is only a particular case of the above type, the values of the altitude being zero and $-18°$. The decl. of the sun on the given date has to be taken from the Nautical Almanac.

Ex. 3.—Find the duration of twilight in lat. 13°·4 N on June 21st.

Fig. 7.

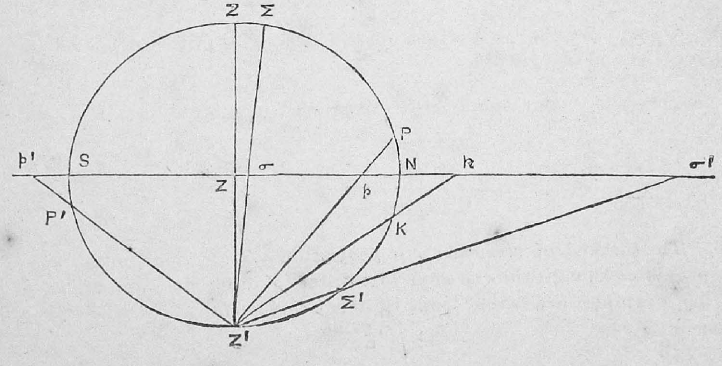
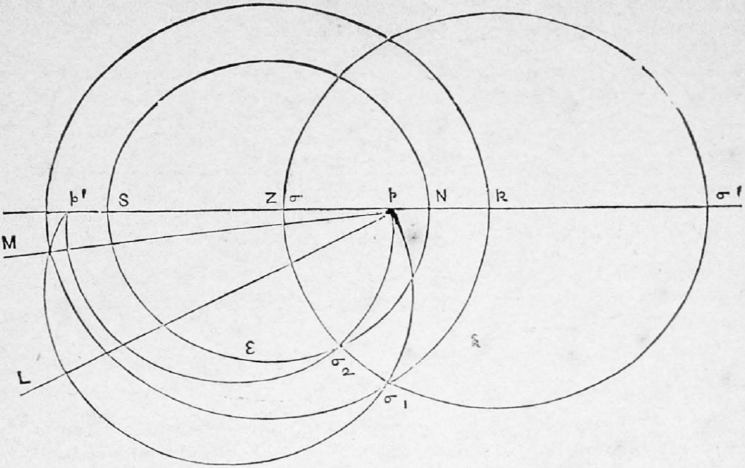


Fig. 8 is the projection on the horizontal plane. NES is the horizon.

Fig. 8.



$k\sigma_1$ is the projection of a parallel to the horizon through K, where ZK [Fig. 7] $=108°$.

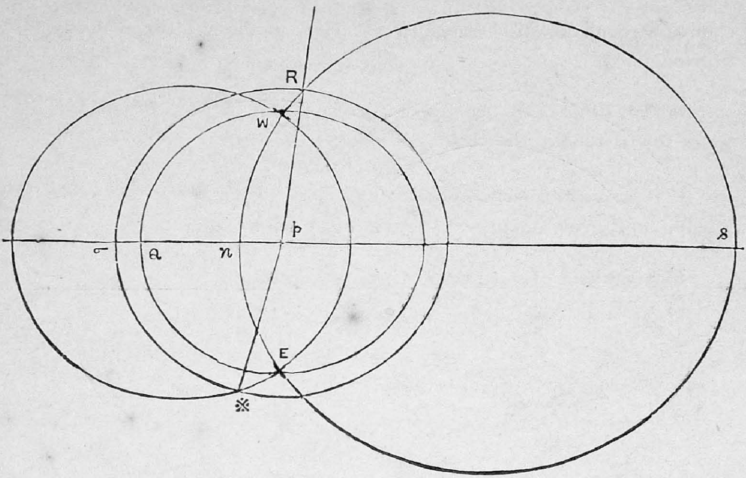
$\sigma\sigma_1\sigma'$ is the projection of the sun's path on June 21st, decl. $=23\frac{1}{2}°$ N.

$p\sigma_1p'$, $p\sigma_2p'$ are the projections of the decl. ⊙ˢ of the sun at the beginning and end of twilight.

$$\begin{aligned} \therefore \quad \text{duration} &= \angle\sigma_1 p \sigma_2 \text{ expressed in time} \\ &= \angle \text{L}p\text{M} \quad \text{,,} \\ &= 20\tfrac{3}{4}° \quad \text{,,} \\ &= 1 \text{ h. } 23 \text{ m.} \end{aligned}$$

6. The method of stereographic projection can also be applied to other problems though the method will differ in different cases. The following examples are taken from Stebbing's *Navigation and Nautical Astronomy.*

Fig 9



Ex. 4.—*Rigel* was on the prime vertical east of the meridian at 8 h. 6 m. 12 s. by chronometer, and on the prime vertical west of the meridian when the same chronometer showed 6 h. 12 m. 24s. Find the latitude of the place and the sidereal time at which *Rigel* will set, given *Rigel's* R. A. 5 h. 9 m. 42 s, decl. 8° 19′S, rate of chronometer 3·8 s. gaining.

Fig. 9 is the proj. on the equatorial plane.

QEW is the equator.

$*\sigma$R is the projection of *Rigel's* daily path.

Make $\sigma p *$ = the hr. ∠ of *Rigel* at the 1st transit across the prime vert., obtained as below :

		h.	m.	s.
	12 h —	8	6	12
i.e.		3	53	48
	+	6	12	24
Interval in mean solar units		10	6	12
Correction for rate of clock				— 1½
∴ hr. ∠		= 5	3	5
∴ hr. ∠ in sid. units		= 5	4	

or 76°.

⊙ *EW is the projection of the prime vert.

⊙ through E, W orthogonal to *EW, is the projection of the horizon.

In Fig. 10 mark off $pn = pn$ in Fig. 9. The ∠ between pP and pN gives the latitude of the place (South).

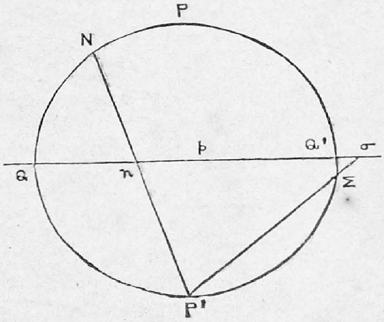
If R be the intersection of the star's daily path and the horizon in Fig. 9, ∠npR gives the hr. ∠ at setting $= 96\frac{1}{2}° = 6$ h. 26 m.

Sidereal time of setting = R.A + hr. ∠ at setting,

= 5 h. 9 m. 42 s. + 6 h. 26 m.

= 11 h. 35 m. 42 s.

Fig. 10.



Ex. 5. At a place in lat 28° N, the sun and *Sirius* (R. A. 6h. 40 m 44 s, decl. 16° 35′S.) rose at the same time. Find the sun's R. A.

Fig. 11.

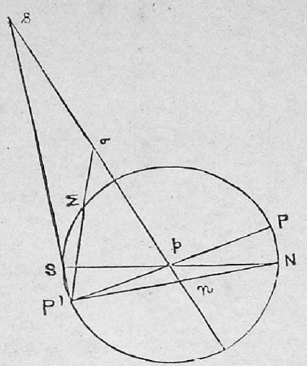


Fig. 12.

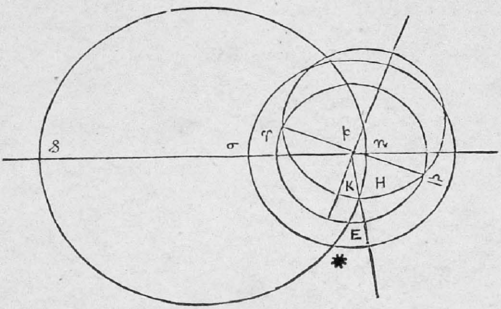


Fig. 11 is the celestial sphere, and Fig. 12 is the proj. on the equatorial plane. The two figures have had to be *reduced* to suit the size of the Journal.

♈ E ♎ is the Equator.

n E s is the proj. of the horizon.

σ *is the daily path of *Sirius*.

* is the position of *Sirius* at rising.

Measure ∠ * *p* ♈ = 100° 11′ (R. A. of *Sirius*.)

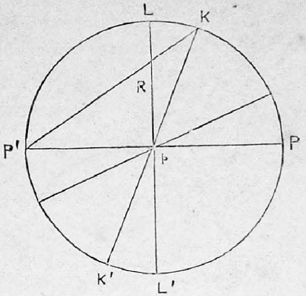
This gives the position of ♈, and also of ♎.

Now, to obtain the proj. of tbe ecliptic :

In the celestial sphere, take L *p* L′ a diameter perp. to PP′ and ♈♎, Fig. 13, and make ∠ L *p* K = $23\frac{1}{2}$°. Join P′K.

OFFICE OF THE REGISTRAR OF BOOKS, REGISTERED 7 JUL 1913 MADRAS.

Fig. 13.



In Fig. 12, take a diameter of the equator perp. to ♈ ♎, and mark off $pk = pk$ in Fig. 13.

⊙ ♈ k ♎ is the proj. of the ecliptic.

H is the position of the sun at rising.

Then the sun's R.A. is ∠HP ♈ = 121°.

161360
600 008.
JL
Bm 21, No 9 IM
No 9.5

REVIEWS.

Essai de Géométrié Analytique Modulaire a Deux Dimensions par Gabriel Arnoux ; pp. xii—160. Librairie Gauthier-Villars 6 *fr.*

This work, which is the fruit of the combined labours of Messrs. Laisant, Tarry and Arnoux, is perfectly original. The learned authors have endeavoured to lay, so to say, the foundations of a new kind of Geometry which blends together the resources of Arithmetic, Algebra and Geometry, and which bids fair to add important results to the field of science.

As far as possible the methods of Analytical Geometry are applied to the elements that come under consideration. But great difficulties arise from the very notion of modularisation or congruence which is fundamental in this treatise. For, equalities are transformed into congruences, and we have, for instance, with modulus 13, the series of congruences $14 = 40 = -12 \ldots = 1$; and the quantities which are operated upon, instead of being continuous as in ordinary Analytical Geometry, are *discontinuous*. In fact, the fundamental operations are performed on *integral numbers*.

In order to simplify this first attempt in the building up of a Modular Analytical Geometry, the field of operation has been limited. The authors have strictly adhered to their set purpose of considering only modular space of two dimensions and *prime* moduli. In spite, however, of these restrictions, the reader will still be startled at the sight of many a wonderful result.

The geometrical operations that are performed in modular space yield, indeed, results which are apparently altogether at variance with our notions of euclidean geometry. Take, for instance, the various points of a stra.ght line in modular geometry and join them two by two. Instead of a single line, a series of lines interesting each other at different angles will be obtained ; and however, all these lines must be considered as having the same slope. Ocular conclusions are altogether deceptive. Greater difficulty still is felt in dealing with the angle between any two segments.

In fact, the subject is quite new and original and abounds with paradoxical results. The first two chapters deal with the notions of point, direction, line, angle, distance and trigonometrical functions ;

and the remaining chapters are concerned with the properties of the straight line and the conic sections.

Clearness and precision mark the whole work. Great patience is needed on the part of the reader in order to master the fundamental notions; but when the first difficulties have once been overcome, the reader is happy to discover undeniable truths under the cloak of apparently paradoxical propositions and finds his efforts well repaid.

C. P.

---

1. *Elementary Plane Trigonometry.* By Hobson and Jessop. (Cambridge University Press. 1910.)

2. *Elementary Trigonometry.* By F. T. Swanwick, M. A., (Cambridge University Press. 1911).

1. In this book, those parts of the subject are treated, which are usually taught in Schools and Intermediate College classes. The method of treatment does not differ essentially from that usually followed. One or two exceptions may be noted. The subject of logarithms and their practical application to the solution of triangles are relegated to the end of the book. One would like to see them given a prominent place in the earlier pages of a book intended for beginners. Chapters XII and XIV contain some useful theorems on the limits of sin θ, etc., and on the rule of proportional parts as applied to the calculation of logarithms, etc. These would enable students to obtain correct notions of the subjects discussed and the authors may to be congratulated on having introduced them into their text-book.

2. The plan of this work is novel in some important respects. The book is divided into three parts. The first part deals with the trigonometrical ratios of *acute* angles and their applications. In the second part these notions are extended so as to be applicable to *obtuse* angles, and in this connection many of the well-known properties of triangles are deduced and are applied to the solution of triangles and to question on heights and distances. In the third part the subject is begun afresh in a general manner, and here we find general proofs of the necessary formulæ; graphs of circular and inverse circular functions are considered in this part.

Certain special features of this novel plan are specially to be commended. The author's preliminary chapter is welcome containing, as it does, an excellent treatment of "Approximate Arithmetic." The statement on page 25 "that any trigonometrical ratio of an angle is equal to the *co-ratio* of its *co-angle*" sums up in an elegant and concise form the relations between the trigonometrical ratios of complementary angles. Chapter VIII concludes with a useful little para on "dimensions." The proofs of the "addition formulæ" at the beginning of Chapter IX seem to be interesting.

The method of dealing with Trigonometry which the author has adopted in parts I and II has an air of undue rigidity and artificiality about it. A common treatment of acute and obtuse angles would have been more reasonable and useful. The necessary and unavoidable consequence of the author's novel treatment has been not only that the subject is begun afresh in each succeeding part, but that, in applying his formulæ to practical examples, the author has to consider, as separate cases, figures which differ from each other owing to angles which were acute in one being obtuse in others. Indian students generally take up the subject when they enter on university courses of study and when their notions of straight lines, angles, etc., are tolerably well advanced. For such students, a more general treatment of trigonometrical ratios might have been of great practical value.

Though the book under review cannot be recommended as a text-book for Indian Schools and Colleges, we hope, that the bold plan of the author and the many commendable features about the book will make it a useful addition to the standard literature on Elementary Trigonometry.

M. V. A.

SHORT NOTES.

Irregular Numbers.

Let a_2, a_3 a_5, a_7,...denote numbers less than unity, where the subscripts 2, 3, 5, 7,...are the series of prime numbers. Then

$$\frac{1}{1-a_2}\cdot\frac{1}{1-a_3}\cdot\frac{1}{1-a_5}\ldots\text{\&c}=1+a_2+a_3+a_2\cdot a_2+a_5+$$
$$a_3a_2+a_7+a_2\cdot a_2.a_2+a_3.a_3+\ldots, \qquad (1)$$

the terms being so arranged that the products obtained by multiplying the subscripts are the series of natural numbers 2, 3, 4, 5, 6, 7, 8, 9, &c.

The above result is easily got if we remember that the natural numbers are formed by multiplying primes and their powers.

2. Similarly, we have

$$\frac{1}{1+a_2}\cdot\frac{1}{1+a_3}\cdot\frac{1}{1+a_5}=1-a_2-a_3+a_2\cdot a_2-a_5+a_2.a_3$$
$$-a_7-a_2.a_2a_2+a_3.a_3.\ \text{\&c.}, \qquad (2)$$

where the sign is negative whenever a term contains an *odd* number of prime subscripts.

3. Put $a_2=1/2^n$, $a_3=1/3^n$, $a_5=1/5^n$, &c., in (1) and we get

$$\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\ldots=\frac{1}{S_n} \quad \ldots \quad (3)$$

where S_n denotes $1/1^n+1/2^n+1/3^n+1/4^n+\ldots$

Changing n into $2n$ in (3) and dividing by the original, we obtain

$$\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\ldots=\frac{S_n}{S_{2n}} \quad \ldots \quad (4)$$

Examples: (i) $\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\ldots\ldots\ldots=\frac{15}{\pi^2} \quad \ldots \quad (5)$

(ii) $\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^4}\right)\ldots\ldots\ldots=\frac{105}{\pi^4} \quad \ldots \quad (6)$

since $S_2=\pi^2/6$, $S_4=\pi^4/90$, $S_8=\pi^8/9450$.

4. Subtract (2) from (1) and put $a_2=2^{-n}$ &c.; then

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{11^n}+\frac{1}{12^n}+\ldots\ldots\ldots\frac{S_n^2-S_{2n}}{2\cdot S_n}$$

where the numbers 2, 3, 5, 7, 8, ...contain an *odd* number of prime divisors.

Examples: (i) $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}+\ldots\ldots\ldots=\frac{\pi^2}{20} \quad \ldots \quad (7)$

(ii) $\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{8^4}+\ldots\ldots\ldots=\frac{\pi^4}{1260} \quad \ldots \quad (8)$

5. Again, (2, 3, 5, 7.........being the prime uumbers)

$$(1+a_2)(1+a_3)(1+a_5)(1+a_7)\text{.........}=1+a_2+a_3+a_5+a_2.a_3+a_7$$
$$+a_2a_5+a_{11}+a_{73}+ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (9)$$

where the product of the subscripts in any term is a natural number containing *dissimilar* prime divisors; and

$$(1-a_2)(1-a_3)(1-a_5)(1-a_7)\text{......}=1-a_2-a_3-a_5+a_2 a_3-a_7 \quad \dots \quad (10)$$

where the signs are negative whenever the number of factors is odd.

6- Replacing as before, a_2 a_3, a_5 ......by the values given in §3 and by using (4), we deduce that

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{.........}=\frac{S_n}{S_{2n}}, \quad \dots \quad \dots (11)$$

where 2, 3, 5, 6, 7,.........are the numbers containing *dissimilar* prime divisors.

7. Also taking half the difference between (3) and (4)

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}+\frac{1}{19^n}$$
$$+\frac{1}{23^n}+\frac{1}{29^n}+\frac{1}{30^n}+\frac{1}{31^n}+\dots=\frac{S_n^2-S_{2n}}{2\,S_n\cdot S_{2n}} \quad \dots \quad (12)$$

where 2, 3, 5,...are numbers containing an *odd* number of *dissimilar* prime divisors.

Examples: (i) $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{9}{2\pi^2}$ ... ... (13)

(ii) $\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots=\frac{15}{2\pi^4}$ ... ... (14)

8. Subtracting (11) from S_n, we have

$$\frac{1}{4^n}+\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{12^n}+\dots=\frac{S_n\,(S_{2n}-1)}{S_{2n}}, \quad \dots \quad (15)$$

where 4, 8, 9,...are composite numbers having *at least* two *equal* prime divisors.

12th May 1913. S. RAMANUJAN.

(*To be continued.*)

The Face of the Sky for July and August 1913.

Sidereal time at 8 p.m.

	July.				*August.*		
	H.	M.	S.		H.	M.	S.
1	14	36	7	...	16	38	20
8	15	3	43	...	17	5	56
15	15	31	19	...	17	33	32
22	15	58	55	...	18	1	8
29	16	26	31	...	18	28	44

From this table the constellations visible during the evenings of July and August can be ascertained by a reference to their position as given in a star-atlas.

Phases of the Moon.

	July.					*August.*			
	D.	H.	M.			D.	H.	M.	
New Moon ...	4	10	36	A.M.	...	2	6	28	P.M.
First Quarter ...	11	3	7	,,	...	9	9	33	A.M.
Full Moon ...	18	11	36	,,	...	17	1	57	,,
Last Quarter ...	26	3	29	P.M.	...	25	5	48	,,

Planets.

Mercury is stationary on July 20 and is in inferior conjunction with the sun on August 4 and is again stationary on August 14. It attains its greatest elongation (18° 25′W) on August 22. It in is conjunction with the moon on August 2 and August 30.

Venus is a morning star. It attains greatest elongation (45° 44′ W) on July 4. It is in conjunction with the moon on July 30 and Aug. 29 at 5·13 A.M. and with Saturn on July 22.

Mars is in conjunction with the moon on July 28 and with Saturn on August 24.

Jupiter is in opposition to the sun on July 5 at 8·30 P.M. It is in conjunction with Jupiter on July 17 and on August 13.

Saturn is in conjunction with the moon on July 2 and August 26.

Uranus is in opposition to the sun on July 29 and is in conjunction with the moon on August 15.

Neptune is in conjuction with the sun on July 19 and with the moon on August 1.

V. Ramesam.

SOLUTIONS.

Question 224.

(M. T. Naraniengar, M.A.) :—If O, H be the circumcentre and the orthocentre of a triangle ABC, the trilinear coordinates of the incentre I are proportional to the geometric means of those of O and H. Prove that a similar relation holds for the normal coordinates of the projections of O, H, and I on any sphere through ABC. Can this be generalized ?

Solution by 'Zero.'

Let $(\alpha\beta\gamma)$ be the *trilinear* coordinates of any point P with respect to the chordal triangle ABC and (x, y, z) the *normal* coordinates of the projection of P on the sphere. Then evidently

$x : y : z =$ ratios of $\perp^{rs}$ from P on the faces of the trihedral solid angle at the centre of the sphere.

$$=\frac{\text{vol.OPBC}}{\triangle\text{OBC}} : \frac{\text{vol.OPCA}}{\triangle\text{OCA}} : \frac{\text{vol.OPAB}}{\triangle\text{OAB}}$$

$$=\frac{\triangle\text{PBC}}{\triangle\text{OBC}} : \frac{\triangle\text{PCA}}{\triangle\text{OCA}} : \frac{\triangle\text{PAB}}{\triangle\text{OAB}}$$

$$=\frac{\alpha\sin\frac{1}{2}a}{\sin a} : \frac{\beta\sin\frac{1}{2}b}{\sin b} : \frac{\gamma\sin\frac{1}{2}c}{\sin c}.$$

$$=\alpha\sec\tfrac{1}{2}a : \beta\sec\tfrac{1}{2}b : \gamma\sec\tfrac{1}{2}c.$$

In other words, the normal coordinates of the projection are proportional to given multiples of the trilinear coordinates of P.

Hence, the result stated in the question is easily verified.

Question 338.

(S. Narayanan, B.A., L.T.) :—"If two pairs of opposite vertices of any complete quadrilateral be conjugate for a conic, so also is the third pair"— Hesse's Theorem.

Prove this from elementary considerations, without employing the methods of reciprocation or invariants.

Solution by N. B. Pendse, M.A., L.L.B.

Let LMN be the triangle formed by the diagonals; and P, Q, R the middle points of AA,′ BB′, CC,′ Let the conic cut AA′, BB′, CC′ in points XX′, YY′, ZZ′, respectively. Suppose XX′, YY′ cut AA′, BB′ harmonically. Then ZZ′ will also cut CC′ harmonically.

For $$\text{PN}\cdot\text{PM} = \text{PX}\cdot\text{PX}'.$$

$$\therefore \quad \frac{\text{PN}}{\text{PX}} = \frac{\text{PX}'}{\text{PM}}, \text{ also } \frac{\text{PN}}{\text{PX}'} = \frac{\text{PX}}{\text{PM}}.$$

$$\therefore \quad \frac{NX}{PX}=\frac{X'M}{PM'}, \text{ and } \frac{NX'}{PX'}=\frac{XM}{PM}.$$

$$\therefore \quad \frac{NX\cdot NX'}{MX\cdot MX'}=\frac{PX\cdot PX'}{PM^2}=\frac{PN\cdot PM}{PM^2}=\frac{PN}{PM}$$

Similarly $\quad QN\cdot QL=QY\cdot QY'.$

$$\therefore \quad \frac{LY\cdot LY'}{NY\cdot NY'}=\frac{QL}{QN}.$$

$$\therefore \quad \frac{NX\cdot NX'\cdot LY\cdot LY'}{MX\cdot MX'\cdot NY\cdot NY'}=\frac{PN}{PM}\cdot\frac{QL}{QN}$$

But
$$\frac{NX\cdot NX'\cdot LY\cdot LY'\cdot MZ\cdot MZ'}{MX\cdot MX'\cdot NY\cdot NY'\cdot LZ\cdot LZ'}=1,$$

since $XX'YY'ZZ'$ are on a cenic ; and

$$\frac{PN\cdot QL\cdot RM}{PM\cdot QN RL}=1, \text{ PQR being collinear.}$$

$$\therefore \quad \frac{MZ\cdot MZ'}{LZ LZ'}=\frac{RM}{RL}.$$

Now, if Z'' be the conjugate of Z for CC', then

$$RZ\cdot RZ''=RM\cdot RL.$$

$$\therefore \quad \frac{MZ\cdot MZ''}{LZ\cdot LZ''}=\frac{RM}{RL}.$$

$$\therefore \quad \frac{MZ'}{LZ'}=\frac{MZ''}{LZ''}.$$

Hence Z' & Z'' coincide. That is, Z,Z' cut CC' harmonically.

Question 379.

(A. C. L. Wilkinson, M. A.) :—Show that the property of a hyperbola that the portion of the tangent at any point intercepted between the asymptotes is bisected at the point of contact and makes a triangle of constant area with the asymptotes, can be generalized for three dimensions as follows :—

Any triangle such that its sides touch a hyperboloid of two sheets and its vertices lie on the asymptotic cone is such that (1) the points of contact with the hyperboloid are the middle points of the sides, (2) the volume of the tetrahedron formed by the triangle and centre of the hyperboloid is constant.

Shew also that if PQR is such a triangle, and O is the centre of the hyperboloid, then $OP^2+OQ^2+OR^2-PQ^2-PR^2-QR^2$ is the algebraic sum of the squares of the axes of the hyperboloid.

Solution by J. C. Swaminarayan, M. A.

As the points P, Q, R lie on the asymptotic cone

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

their coordinates may be written (λ, α), (μ, β) (ν, γ) where (λ, α)... denote $(a\lambda, b\lambda \cos\alpha, c\lambda \sin\alpha)$,...

If the middle point of QR lies on the hyperboloid, we have

$$(\mu+\nu)^2-(\mu \cos \beta+\nu \cos \gamma)^2-(\mu \sin \beta+\nu \sin \gamma)^2=4,$$

or $$2 \mu\nu (1-\cos \beta \cos \nu-\sin \beta \sin \nu)=4,$$

or $$\mu\nu\sin^2\frac{\beta-\gamma}{2}=1$$

$$\therefore \quad \mu\nu=\operatorname{cosec}^2\frac{\beta-\gamma}{2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

In this case, QR can easily be shown to lie in the tangent plane to the hyperboloid, at the middlepoint of QR, namely:

$$\frac{x}{a}\frac{\mu+\nu}{2}-\frac{y}{b}\cdot\frac{\mu\cos\beta+\nu\cos\gamma}{2}-\frac{z}{c}\cdot\frac{\mu\sin\beta+\nu\sin\gamma}{2}=1 \qquad (2)$$

The condition that QR should lie in plane (2) is

$$(\mu^2-\nu^2)-(\mu^2 \cos^2 \beta-\nu^2 \cos^2 \gamma)-(\mu^2 \sin^2 \beta-\nu^2 \sin^2 \gamma)=0,$$

which is obvious.

Hence, by (1) the sides of PQR will touch the hyperboloid.

Next, the volume of the tetrahedron OPQR

$$=\frac{1}{6}abc\lambda\mu\nu \begin{vmatrix} 1, & \cos\alpha, & \sin\alpha \\ 1, & \cos\beta, & \sin\beta \\ 1, & \cos\gamma, & \sin\gamma \end{vmatrix}$$

$$=\frac{1}{6}\, abc\, \lambda\mu\nu\Sigma\sin(\gamma-\beta)$$

$$=\frac{2}{3}\, abc\, \lambda\mu\nu \prod \sin\frac{\beta-\gamma}{2}$$

$$=\frac{2}{3}\, abc \text{ by (1).}$$

Lastly $\quad OP^2+OQ^2+OR^2-QR^2-RP^2-PQ^2$

$$=a^2\{2\mu\nu+2\nu\lambda+2\lambda\mu-\lambda^2-\mu^2-\nu^2\}$$
$$+b^2\{2\Sigma\mu\nu \cos \beta \cos \gamma-\Sigma\lambda^2 \cos^2\alpha\}$$
$$+c^2\{2\Sigma\mu\nu \sin \beta \sin \gamma-\Sigma\lambda^2 \sin^2 \alpha\}.$$

Now by (1) $\quad \lambda=\sin\frac{\beta-\gamma}{2}\operatorname{cosec}\frac{\gamma-\alpha}{2}\operatorname{cosec}\frac{\alpha-\beta}{2}$

$$\mu=\sin\frac{\alpha-\gamma}{2}\operatorname{cosec}\frac{\alpha-\beta}{2}\operatorname{cosec}\frac{\beta-\gamma}{2}$$

$$\nu=\sin\frac{\alpha-\beta}{2}\operatorname{cosec}\frac{\beta-\gamma}{2}\operatorname{cosec}\frac{\gamma-\alpha}{2}.$$

Hence the coefficient of a^2 is

$$\Sigma 2\,\mu\nu - \Sigma\lambda^2 = \left(\lambda^{\frac{1}{2}}+\mu^{\frac{1}{2}}+\nu^{\frac{1}{2}}\right) \prod \left(\mu^{\frac{1}{2}}+\nu^{\frac{1}{2}}-\lambda^{\frac{1}{2}}\right)$$

$$= \frac{\left(\Sigma \sin\frac{\beta-\gamma}{2}\right) \prod \left\{\sin\frac{\gamma-\alpha}{2}+\sin\frac{\alpha-\beta}{2}-\sin\frac{\beta-\gamma}{2}\right\}}{\sin^2\frac{\beta-\gamma}{2}\sin^2\frac{\gamma-\alpha}{2}\sin^2\frac{\alpha-\beta}{2}}$$

$$= \frac{256 \prod \left\{\sin^2\frac{\beta-\gamma}{4}\cos^2\frac{\beta-\gamma}{4}\sin^2\frac{\gamma-\alpha}{2}\cos^2\frac{\gamma-\alpha}{4}\sin^2\frac{\alpha-\beta}{4}\cos^2\frac{\alpha-\beta}{4}\right\}}{\sin^2\frac{\beta-\gamma}{2}\sin^2\frac{\gamma-\alpha}{2}\sin^2\frac{\alpha-\beta}{2}}$$

$=4.$

Denote by B, C the coefficients of b^2 and c^2. Then

$$4-B-C=\Sigma\ 2\,\mu\nu\,(1-\cos\beta\ \cos\gamma-\sin\beta\ \sin\gamma) -\Sigma\lambda^2(1-\cos^2\alpha-\sin^2\alpha).$$

$$=4+4+4=12.$$

Now

$$\sqrt{\lambda e^{i\alpha}}+\sqrt{\mu e^{i\beta}}+\sqrt{\nu e^{i\gamma}}$$

$$= \frac{\sum \sqrt{\sin^2\frac{\beta-\gamma}{2}}\ (\cos\alpha+i\,\sin\alpha)}{\sqrt{\prod \sin\frac{\beta-\gamma}{2}}}$$

$$= \frac{\sum \left\{\sin\frac{\beta-\gamma}{2}\ \left(\cos\frac{\alpha}{2}+i\ \sin\right)\right\}}{\sqrt{\prod \sin\frac{\beta-\gamma}{2}}}$$

$$=0$$

That is, $\Sigma\ 2\ \mu\nu e^{i\,(\beta+\gamma)} - \Sigma\lambda^2 e^{2i\alpha}=0.$

$\therefore$ $\Sigma\ 2\ \mu\nu\ \{\cos\,(\beta+\gamma)+i\ \sin\,(\beta+\gamma)\}$
$-\Sigma\lambda^2\ \{\cos\ 2\ \alpha+i\ \sin\ 2\ \alpha\}=0.$

Equating the real parts, we get

$$\Sigma\ 2\ \mu\nu\ \cos\,(\beta+\gamma)-\Sigma\lambda^2\ \cos\ 2\ \alpha=0.$$

Thus $B-C=0.$

But $4-B-C=12.$

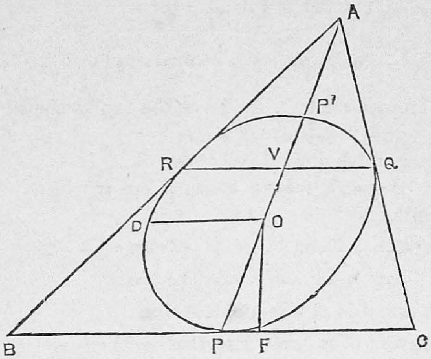
$\therefore$ $B=C=4.$

Hence $\Sigma OP^2-\Sigma QR^2=4\ (a^2-b^2-c^2)$

Question 403.

(S. Narayanan, B. A., L.T.) :—Shew that the square of the semi-axis of the maximum inscribed ellipse of a triangle is $\frac{1}{9}\Delta\cot\ \lambda$, where Δ is the area of the triangle and λ the semi-steiner angle, (the smaller corresponding to the major axis and the greater to the minor axis).

Solution (1) *by R. Tata, M.A. and* (2) *by V. B. Naik, M.A.*



(1) Let ABC be the triangle, O (centroid) the centre of the maximum inscribed ellipse, P,Q,R (mid-points of sides) the points of contact, OF ⊥ BC, OD the semi-diameter parallel to BC, and V the middle point of QR.

From the ellipse $\quad OD^2 : OP^2 = QV^2 : PV \cdot VP'$

From the triangle $\quad OP = \frac{1}{3}m;\ PV = \frac{1}{2}m,\quad P'V = \frac{1}{6}m,\quad QV = \frac{1}{4}a$

$$\therefore \qquad OD^2 = \frac{OP^2 . QV^2}{PV \cdot P^1V}.$$

$$= \frac{a^2}{12}.$$

Also $\quad OP^2 = \frac{1}{9}m^2 = \{\ 2(b^2+c^2)-a^2\ \}/36.$

If the semi-axes be r_1, r_2

$$r_1^2 + r_2^2 = OP^2 + OD^2 = \frac{2(b^2+c^2)-a^2}{36} + \frac{a^2}{12}$$

$$= \frac{a^2+b^2+c^2}{18}.$$

Again, from the ellipse $r_1 \cdot r_2 = \text{OF} \cdot \text{OD}$

$$= \frac{2\triangle}{3a} \cdot \frac{a}{2\sqrt{3}}$$

$$\therefore \qquad r_1^2 r_2^2 = \frac{\triangle^2}{27}.$$

Thus the squares of the semi axes are the roots of the equation

$$x^2 - \frac{x}{18}(a^2+b^2+c^2) + \frac{\triangle^2}{27} = 0,$$

i.e. $$x^2 - \frac{2}{9}\triangle x \cot \omega + 3 \cdot \frac{\triangle^2}{81} = 0. \qquad \dots \qquad \dots \qquad \text{(i)}$$

since $\cot \omega = \dfrac{a^2+b^2+c^2}{4\triangle}$, ω being the Brocard angle of the triangle.

For the semi-steiner angle λ we have the fundamental equation

$$\sin(2\lambda + \omega) = 2 \sin \omega,$$

which gives $$\cos 2\lambda + \sin 2\lambda \cot \omega = 2,$$

$$2\cos^2 \lambda + 2 \sin \lambda \cos \lambda \cot \omega = 3.$$

Dividing by $\sin^2 \lambda$,

$$2\cot^2 \lambda + 2 \cot \lambda \cot \omega = 3 \operatorname{cosec}^2 \lambda.$$

$$\therefore \qquad \cot^2 \lambda - 2 \cot \lambda \cot \omega + 3 = 0 \qquad \dots \qquad \dots \qquad \text{(ii)}$$

Comparing (i) and (ii) the result follows.

(2) The equation of a conic inscribed in ABC is

$$\sqrt{(\text{L}\alpha)} + \sqrt{(\text{M}\beta)} + \sqrt{(\text{N}\gamma)} = 0.$$

Its area is proportional to $\dfrac{\text{LMN}}{(bc\text{L}+ca\text{M}+ab\text{N})^3}$, which is a maximum when $\text{L} : \text{M} : \text{N} = a : b : c$; *i.e.* when the conic touches the sides of the triangle at their middle points. The centre of the conic is therefore the centroid of the triangle.

If α, β, γ and α', β', γ' are the coordinates of the foci of the conic, the centroid is the middle point of their join, so that

$$\tfrac{1}{2}(\alpha + \alpha') = \frac{2\Delta}{3a}, \qquad \dots \qquad \dots \qquad \dots \qquad (1)$$

$$\tfrac{1}{2}(\beta + \beta') = \frac{2\Delta}{3b}, \qquad \dots \qquad \dots \qquad \dots \qquad (2)$$

$$\tfrac{1}{2}(\gamma + \gamma') = \frac{2\Delta}{3c} \qquad \dots \qquad \dots \qquad \dots \qquad (3)$$

Also, $\alpha\alpha' = \beta\beta' = \gamma\gamma'$

= product of perpendiculars from foci on tangents

$= k^2$,

where k^2 is the square of the semi-major or semi-minor axis, according as the foci are imaginary or real points.

We have therefore, from (1), (2) and (3)

$$\alpha^2-\frac{4\Delta}{3a}\alpha+k^2=0, \quad \dots \quad \dots \quad \dots \quad (4)$$

$$\beta^2-\frac{4\Delta}{3b}\beta+k^2=0, \quad \dots \quad \dots \quad \dots \quad (5)$$

$$\gamma^2-\frac{4\Delta}{3c}\gamma+k^2=0. \quad \dots \quad \dots \quad \dots \quad (6)$$

Also $\quad a\alpha+b\beta+c\gamma=2\Delta \quad \dots \quad \dots \quad \dots \quad (7)$

Solving (4) $\quad a\alpha=\frac{2\Delta}{3}\pm\frac{1}{3}\sqrt{(4\Delta^2-9a^2k^2)}$

Similarly, solving (5) and (6) and substituting in (7) we have $\sqrt{(4\Delta^2-9a^2k^2)}+\sqrt{(4\Delta^2-9b^2k^2)}+\sqrt{(4\Delta^2-9c^2k^2)}=0 \quad \dots \quad (8)$

Let us suppose that $k^2=\frac{1}{9}\Delta x$, where x is unknown.

Then $\quad \sqrt{(4\Delta-a^2x)}+\sqrt{(4\Delta-b^2x)}+\sqrt{(4\Delta-c^2x)}=0$

$\therefore \quad 4\Delta-a^2x=4\Delta-b^2x+4\Delta-c^2x+2\sqrt{\{16\Delta^2-4\Delta(b^2+c^2)x+b^2c^2x^2\}}$

$\therefore \quad (2bc\cos Ax-4\Delta)^2=64\Delta^2-16\Delta(b^2c^2)x+4b^2c^2x^2.$

$\therefore \quad 4b^2c^2\sin^2 A\, x^2\times16\times\Delta(bc\cos A-b^2-c^2)+48\Delta^2=0$

$$\therefore \quad x^2-x\frac{a^2+b^2+c^2}{2\Delta}+3=0.$$

$$\therefore \quad x^2-2x\cot\omega+3=0.$$

where ω is the Brocard angle of the triangle.

Now the semi-steiner angles are given by the equation

$$\sin(\omega+2\lambda)=2\sin\omega.$$

i.e., $\quad \cos 2\lambda+\cot\omega\sin 2\lambda=2.$

Put $\cot\lambda=x$, this becomes

$$x^2-2x\cot\omega+3=0$$

$\therefore$ The square of a semi-axis of the conic is $\frac{1}{2}\Delta x$ when x is the cotangent of the semi-steiner angle, the smaller evidently corresponding to the major axis, and the larger to the minor axis.

Additional solution by J. C. Swaminarayan, M.A.

Questions 404 and 436.

404. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—In a tetrahedron for which the products of the pairs of opposite edges are equal, prove that any plane parallel to the tangent plane to the circumscribing sphere at one of the vertices, cuts the three edges through that vertex in an equilateral triangle.

436. (N. DURAI RAJAN, B.A.) :—Through a point P within a tetrahedron planes are drawn parallel to the four faces. Shew that the twelve points of intersection of these planes and the edges will lie on a sphere, if BC.AD = CA·BD = AB.CD. (This is the analogue of the Lemoine circle).

Solution by A. C. L. Wilkinson, M.A., F.R.A.S., with remarks.

(1) With respect to any tetrahedron ABCD, consider the system of spheres through BCD, given by the equation :

$$\Sigma AB^2.\alpha\beta+\lambda\alpha(\alpha+\beta+\gamma+\delta)=0.$$

Where this meets AD or $\beta=0$, $\gamma=0$, we have

$$AD^2,\ \alpha\delta+\lambda\alpha(\alpha+\delta)=0,$$

whence the three points where the above sphere meets the three edges through A are :

$$(AB^2+\lambda,-\lambda,0,0),\ (AC^2+\lambda,0,-\lambda,0),\ (AD^2+\lambda,0,0-\lambda).$$

The plane through these three points is easily found to be

$$\lambda(\alpha+\beta+\gamma+\delta)+AB^2.\beta+AC^2.\gamma-AD^2.\delta=0,$$

whence for different values of λ these planes are parallel to the tangent plane to the circumscribing sphere at A. We may call these planes *antiparallels* to the face BCD.

Consider now the locus of the centroids of the triangles in which these antiparallel planes cut the edges through A. The centroid is

$$\left\{\frac{1}{3}\left(3+\frac{\lambda}{AB^2}+\frac{\lambda}{AC^2}+\frac{\lambda}{AD^2}\right),\ -\frac{1}{3}\frac{\lambda}{AB^2},\ -\frac{1}{3}\frac{\lambda}{AC^2},\ -\frac{1}{3}\frac{\lambda}{AD^2}\right\}$$

and the line joining this to (1,0,0,0) is

$$\frac{\alpha-1}{\frac{1}{AB^2}+\frac{1}{AC^2}+\frac{1}{AD^2}}=\frac{\beta}{-\frac{1}{AB^2}}=\frac{\gamma}{-\frac{1}{AC^2}}-\frac{\delta}{-\frac{1}{AD^2}}.$$

Similarly the corresponding locus through B is :

$$\frac{\alpha}{-\frac{1}{AB^2}}=\frac{\beta-1}{\frac{1}{AB^2}+\frac{1}{BC^2}+\frac{1}{BD^2}}=\frac{\gamma}{-\frac{1}{BC^2}}=\frac{\delta}{-\frac{1}{BD^2}}.$$

If these lines intersect, by comparing the values of γ and δ, we have AC·BD = AD·BC.

Hence in the tetrahedron for which AB·CD = AC·BD = AD·BC, *the four lines joining the vertices to the controids of the triangles formed by the corresponding antiparallels are concurrent.*

We may call this point of concurrence the *symmedian* point of this special tetrahedron, its coordinates may be expressed symmetrically as:

(BC.CD.DB, AC.CD.DA, AB·BD·DA, AB.BC CA).

(2) *The triangles formed by the antiparallel planes of this special tetrahedron are equilateral.*

The distance between $\left(1+\frac{\lambda}{AB^2}, -\frac{\lambda}{AB^2},0,0\right)$ & $\left(1+\frac{\lambda}{AC^2}0, -\frac{\lambda}{AC^2},0\right)$ is

$$-D^2=\Sigma AB^2(\alpha-\alpha')(\beta-\beta')=-\frac{\lambda^2 BC^2}{AB^2.AC^2},$$

whence $$D=\pm\frac{\lambda.BC\cdot AD}{AB\cdot AC\cdot AD}.$$

Since this expression is symmetrical with respect to A, the theorem of Q. 404 follows.

(3) *The 12 points where the antiparallel planes through the symmedian point intersect the corresponding edges of the tetrahedron lie on a sphere whose centre is at the symmedian point.*

The symmedian point is the centroid of each of the four equilateral triangles, it is only necessary therefore to show that the dimensions of all the triangles are the same.

We require the special values of λ and D.

The plane $\lambda(\alpha+\beta+\gamma+\delta)+AB^2.\beta+AC^2.\gamma+AD^2.\delta=0$ passes through the symmedian point (BC.CD.DB,...,...,...,), if

$$\lambda\Sigma BC.CD.DB+AB.AC.AD(AB.CD+AC.BD+AD.BC)=0,$$

substituting this value of λ we see that D is the same for all the four triangles, whence the theorem follows.

(4) To prove Q. 436, consider any point $(\alpha_1,\beta_1,\gamma_1,\delta_1)$, write $\alpha_1+\beta_1+\gamma_1+\delta_1=k$.

The plane parallel to $\alpha=0$ is $\frac{\alpha}{\alpha_1}=\frac{\alpha+\beta+\gamma+\delta}{\alpha_1+\beta_1+\gamma_1+\delta_1}$ and where this meets $\beta=0$, $\gamma=0$, we have for the point of intersection $(\alpha_1,0,0,k-\alpha_1)$.

Consider the sphere

$$\Sigma AB^2.\alpha\beta+(l\alpha+m\beta+n\gamma+p\delta)(\alpha+\beta+\gamma+\delta)=0,$$

if it passes through the 12 points, we have

$$AD^2.\alpha_1(k-\alpha_1)+k\{l\alpha+p(k-\alpha_1)\}=0$$

and $$AD^2.\delta_1(k-\delta_1)+k\{l(k-\delta_1)+p\delta_1\}=0.$$

Solve for l, divide by the factor $\beta_1+\gamma_1$ and we get

$$-k^2.,l=AD^2.\delta_1(k-\alpha_1)$$
$$=AB^2.\beta_1(k-\alpha_1)$$
$$=AC^2.\gamma_1(k-\alpha_1) \text{ by symmetry.}$$

Whence $AD^2.\delta_1=AB^2.\beta_1=AC^2.\gamma_1$ and three similar relations.

These give $AB.CD=AC.BD=AD.BC$ and further we see that $(\alpha_1,\beta_1,\gamma_1,\delta_1)$ is identical with the symmedian point.

(5) *The centres of the circumscribing sphere of the tetrahedron and the, spheres obtained in (3) and (4) are collinear.*

Take the equation of the sphere (3) as :

$$\Sigma AB^2.\alpha\beta+(l\alpha+m\beta+n\gamma+p\delta)(\alpha+\beta+\gamma+\delta)=0.$$

It passes through the points

$$(AB^2+\lambda_1,-\lambda_1,0,0) \text{ and } (-\lambda_2, AB^2+\lambda_2,0,0)$$

where λ_1, λ_2 are the parameters corresponding to the antiparallels to BCD and ACD through the symmedian point and λ_1 is given by (3).

Hence $$-AB^2.\lambda_1(AB^2+\lambda_1)+AB^2\{l(AB^2+\lambda_1)-m\lambda_1\}=0,$$

and $$-AB^2.\lambda_2(AB^2+\lambda_2)+AB^2\{-l\lambda_2+m(AB^2+\lambda_2)\}=0,$$

sutbract and we get $l-m=\lambda_1-\lambda_2$ and similar expressions. Thus

$$l=\lambda_1+k,\ m=\lambda_2+k,\ n=\lambda_3+k,\ p=\lambda_4+.k.$$

The radical plane of the circumsphere and (3) is therefore

$$\lambda_1\alpha+\lambda_2\beta+\lambda_3\gamma+\lambda_4\delta+k(\alpha+\beta+\gamma+\delta)=0.$$

From (4) we have l proportional to $AD^2.\delta_1(k-\alpha_1)$, or to $S\{k.AB.AC.AD-S^3\}$, where $S=AB.CD=AC.BD=AD.BC$.

Thus the radical plane of the circumsphere and (4) is

$$u(\lambda_1\alpha+\lambda_2\beta+\lambda_3\gamma+\lambda_4\delta)+v(\alpha+\beta+\gamma+\delta)=0,$$

where u, v are constants easily determined.

These two radical planes are parallel and therefore the three centres are collinear.

Question 411.

(A. C. L. Wilkinson, M.A., F.R.A.S.) :—Show that the evolute of the curve whose intrinsic equation is $\frac{s^2}{a^2}-\frac{\rho^2}{b^2}=1$, is $\frac{\rho^2a^2}{b^4}-\frac{s^2}{b^2}=1$. Determine the forms of both these curves. Show that for both the curves a point O exists such that the projection of the radius vector from O on the tangent and normal at any point are proportional to s and ρ

respectively. Examine more closely the meaning of s in the first of these equations.

Solution by J. C. Swaminarayan, M.A., V. B. Naik, M.A., and N. Durai, Rajan, B.A.

We have $$\frac{s^2}{a^2}=1+\frac{1}{b^2}\left(\frac{ds}{d\psi}\right)^2$$

$$\therefore \quad d\psi=\frac{a}{b}\cdot\frac{ds}{\sqrt{s^2-a^2}}$$

$$\therefore \quad \psi=\frac{a}{b}\cosh^{-1}\frac{s}{a},$$

provided $s=a$, when the tangent is parallel to the x—axis

$$\therefore \quad s=a\cosh\frac{b}{a}\psi.$$

The equation of the evolute is $s=b\sinh\frac{b}{a}\psi$,

whence $$\rho=\frac{b^2}{a}\cosh\frac{b}{a}\psi$$

$$\therefore \quad \frac{a^2\rho^2}{b^4}-\frac{s^2}{b^2}=1.$$

In the first of these two curves $\rho=b\sinh\frac{b}{a}\psi$.

$$\therefore \quad (D^2+1)p=b\sinh\frac{b}{a}\psi.$$

$$\therefore \quad p=A\cos\psi+B\sin\psi+\frac{a^2b}{a^2+b^2}\sinh\frac{b}{a}\psi,$$

= projection of theradius vector on the normal.

$$\therefore \quad \frac{dp}{d\psi}=-A\sin\psi+B\cos\psi+\frac{a^2+b^2}{ab^2}\cosh\frac{b}{a}\psi,$$

= projection of the radius vector on the tangent.

The origin can, in general, be so selected as to have the arbitrary constants A and B, each equal to zero. In that case $p\propto\rho$ and $\frac{dp}{d\psi}\propto s$. It is easy to see that similar results hold good for the evolute.

For the forms of the curves see Routh's *Dynamics of a Particle*, pp. 207—8.

Partial solution by N. B. Pendse.

QUESTIONS FOR SOLUTION.

457. (S. P. Singaravelu Modeliar, B.A., L.T.):—Shew that the curve $4(2x^2-1)^2+(y^2-4)(y^2-1)^2=0$ is the path of a point whose motion is the resultant of two simple harmonic motions in perpendicular directions.

458. (K. J. Sanjana, M.A.):—Sum the series—

$$S\equiv\frac{\cos\theta}{1}+\tfrac{1}{2}\frac{\cos^3\theta}{3}+\frac{1\cdot3}{2\cdot4}\,\frac{\cos^5\theta}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{\cos^7\theta}{7}+\dots,$$

$$T\equiv\frac{\cos^2\theta}{2}+\tfrac{2}{3}\frac{\cos^4\theta}{4}+\frac{2\cdot4}{3\cdot5}\,\frac{\cos^6\theta}{6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\,\frac{\cos^8\theta}{8}+\dots;$$

and prove that

$$\tfrac{1}{2}\pi S-T=\frac{4}{\pi}\left\{\frac{\cos\theta}{1^3}-\frac{\cos3\theta}{3^3}+\frac{\cos5\theta}{5^3}-\dots\right\}$$

459. (S. Krishnaswami Aiyangar):—If $a_r\equiv\sum_{1}^{n=\infty}\left(\frac{n^r}{\underline{|r}}\right)$ prove that

$$a_{r+1}=a_r+r.a_{r-1}+\frac{r.r-1}{1\cdot2}\,a_{r-2}+\dots\dots r.a_1+a_0.$$

460. (V. V. Satyanarayana):—Find a point such that the feet of the perpendiculars from it on the sides of a given quadrilateral may form a parallelogram. Discuss the number of solutions.

461. (Selected):—Prove that the dihedral angle of a regular tetrahedron is the supplement of that of a regular octahedron.

462. (V. Ramaswami Aiyar, M. A.):—EF being one of the diagonals of a quadrilateral, if Ep, Ep' be perpendiculars drawn to the sides meeting at F, and Fq, Fq' those drawn to the sides meeting at E, and O be the point of intersection of pp', qq'; let us call O the *Orthocentre*, and the circle, centre O, and square of radius $=Op.Op'=Oq.Oq'$, the *Polar Circle*, of the quadrilateral,—with respect to the diagonal EF.

Prove (1) that the three ortho centres of a quadrilateral lie on a line which contains also the orthocentres of each of the constituent triangles of the quadrilateral; and (2) that the three polar circles of a quadrilateral belong to a coaxal system which contains also the polar circles of each of the constituent triangles of the quadrilateral.

463. (S. RAMANUJAN) :—If $\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}dx = \phi(n)$, then

$$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}}-1}dx = \phi(n) - \frac{1}{2n} + \phi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^3}}.$$

Find $\phi(n)$, and hence shew that

$$\phi(0) = \frac{1}{12}, \quad \phi\left(\frac{\pi}{2}\right) = \frac{1}{4\pi}, \quad \phi(\pi) = \frac{2-\sqrt{2}}{8},$$

$$\phi(2\pi) = \frac{1}{16}, \quad \phi(\infty) = 0.$$

464. (S. RAMANUJAN) :—$(2^n - 7)$ is a perfect square for the values 3, 4, 5, 7, 15 of n. Find other values.

465. (S. NARAYANA Aiyar, M.A.) :—When n lies between (-1) and (-2), shew that

(i) $\int_0^\infty x^{n-1} \log\left(\frac{\sinh x}{x}\right) dx = \frac{\pi^{n+1}}{n \sin \frac{1}{2}\pi n} S_n,$

(ii) $\int_0^\infty x^{n-1} \log \frac{\cosh x - \cos x}{x^2} dx = \frac{2^{\frac{1}{3}n}\pi^{n+1}}{n \sin \frac{1}{4}\pi n} S_n,$

where $S_n = 1^n + 2^n + 3^n + \ldots\ldots\ldots$ to infinity.

466. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—ABCD in a tetrahedron. Planes are drawn through AB, AC, AD perpendicular respectively to the planes ACD, ABD, ABC. Show that they have a common line of intersection.

Show also that each of the four lines thus obtained, one through each vertex, intersects each of the five perpendiculars from the vertices of the tetrahedron on the opposite faces.

If the hyperboloid is described passing through these eight straight lines, prove that the absolute value of the product of the principal axes is equal to the absolute value of

$$\frac{(AB^2+CD^2-AC^2-BD^2)(AC^2+BD^2-AD^2-BC^2)(AD^2+BC^2-AB^2-CD^2)}{24V}$$

where V is the volume of the tetrahedron.

467. (N. SANKARA AIYAR, M.A.) :—If O is the circumcentre of ABC, and A′B′C′ the triangle whose sides bisect OA, OB, OC perpendicularly, prove that AA′, BB′, CC′ are concurrent.